TWO MEANINGS OF THE 'EQUAL' SIGN AND SENSES OF COMPARISON AND SUBSTITUTION METHODS

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In this paper we analyze the meanings of the 'equal' sign as generated by sense production of the methods of substitution and comparison for solving problems and systems of equations of two unknowns. These methods are usually introduced through an extension process of the syntax and meanings recently learned by students in order to solve problems using linear equations with one unknown. Through this process some users were able to confer sense to the methods and thus generate the new meanings required.

In Filloy (1991) we introduced the notions of ***meaning*** and ***sense*** for analyzing the learning processes and the creation of rules which allow to coordinate the actions performed for solving one-unknown problems through "concrete models" (see Filloy and Rojano, 2001; and Filloy, Rojano and Solares, 2002). In this paper we use these notions for studying the transition from one-unknown representation and manipulation to the representation and manipulation of one unknown given in terms of another unknown—In fact, this transition corresponds to a ***didactic cut*** (see Filloy and Rojano, 1989; Solares, 2002). The new representation of the unknown is used in the ***comparison and substitution methods*** in such a way which allows the **reduction** of a two-unknown problem to a one-unknown problem and making possible to apply the previously learned syntax in order to solve one-unknown linear equations. In the particular case of the system $(S_1):\begin{cases} y = 12 - x \\ 5x - 6 = y \end{cases}$ the performance of the ***comparison method*** entails the equalization of two operation chains for one of the unknowns and for the data which allow to calculate the other unknown's value. That is, two ways for calculating the value of one of the unknowns are equalized. And in the case of the system $(S_2):\begin{cases} x + y = 12 \\ 5x - 6 = y \end{cases}$ to perform the ***substitution*** entails replacing the '***y***' in the second equation in the first and through this operation chain find the '***x***' value. Thus, a chain of operations is substituted into another.

We will analyze the meanings of the 'equal' sign generated by children between 13 and 14 years old when using the comparison and substitution methods in two-unknown equations' solving process. For the students interviewed the ***sense*** of this methods is given by the linking of all actions performed. At the beginning of the learning process these action chains are not yet provided of sense. The increasing syntactical complexity of the relations between the data and the unknowns, the changes in the data's or solutions' numerical domains, for example, obstruct both the use of the methods and the spontaneous solution strategies. At that stage, readings from more concrete strata of the new Mathematical Sign System do not allow to identify the changes in the problematic situation as members of the same kind of problems. Only when the sense conferred through the sequence of mathematical texts in the Teaching Model is acquired these strata will be identified as members of the same kind of problems –susceptible of being

solved through the same process, or chain of actions. That is the moment when the new notions –such as the new notion of equality– will become stable (see Matz, 1980; Kieran, 1981; Kieran and Sfard, 1999; Drouhard, 1992).

Here, it is useful to see once again the way in which Mt. (Filloy and Rojano, 1989, pp. 21-22), one of the subjects interviewed, generated the meaning assigned to the 'equal' sign when learning the syntax for solving one-unknown linear equations: **IMt26. 10x – 18 = 4x + 6...**

> **Mt:**…if I obtain the value of ***x*** and I perform that operation (points towards "***10x – 18***") I obtain one result. That result has to be equal to this (points towards "***4x + 6***")

THEORETICAL AND METHODOLOGICAL FRAMEWORK

For the experimental design a **Local Theoretical Model** (Filloy, 1990) was built up in order to explain, upon the semiotic notion of **Mathematical Sign Systems** (MSS) the empirical observations obtained through videotaped clinical interviews.

From the theoretical perspective of the **Local Theoretical Models** each specific object of study is analyzed through four interrelated components: (1) the **Formal Competence Model**; (2) the **Teaching Model**; and (3) the **Cognitive Process** and (4) **Communication Models**. Below, the specific characteristics of these components in our study will be described.

FORMAL COMPETENCE MODEL

In order to construct **the Formal Model** component, we used the syntax model for simple algebraic expressions and equations developed by Kirshner (1987) and completed by Drohuard (1992). Besides, we incorporated the semantic elements proposed by Drohuard (1992) in order to study the meanings of the algebraic writing. These studies on algebraic syntax and semantics render important results for teaching –such as Drouhard's definition of *automathe formel* for defining subjects who center their attention on the rules that have to be applied (*sens*) and not in the results obtained (*dénotation*)–, these studies do not incorporate to the analysis the spontaneous usage that learners give to already learned elements of the algebraic language in order to solve new problems.

The **Formal Competence Model** that we designed allows us to study the syntactical complexity of the algebraic substitution and comparison methods used for solving equation systems. Substitution method results more complex.

TEACHING MODEL

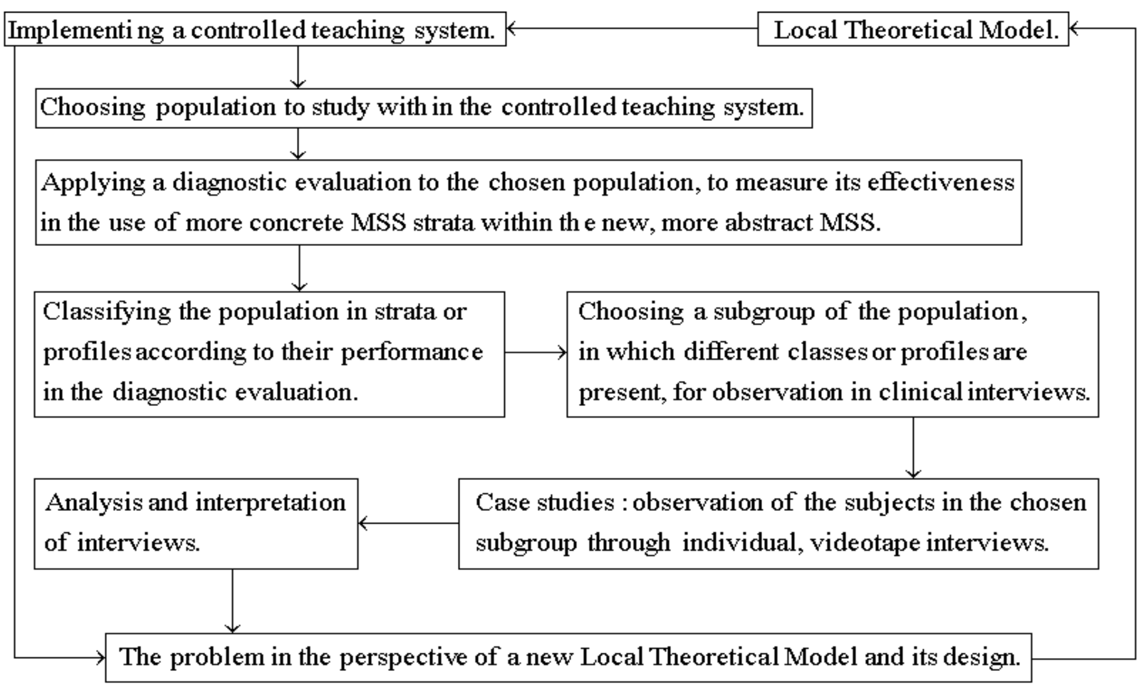
Upon the base of the analysis performed at the formal level we adopted the following **didactical route** for introducing these methods –coming from the previously acquired competencies for solving one-unknown linear equations: (1) reduction of the two-unknown and two-equation system to a one-unknown equation through the application of comparison or substitution; (2) solution of the one-unknown equation applying the previously learned syntax; (3) substitution of the numerical value found in one of the two equations; and (4) solution of the equation through the application of the previously learned syntax.

COGNITIVE PROCESSES MODEL

For the definition of the **Cognitive Processes Model** we used the cognitive tendencies list provided by Filloy (1991). In the current study the following items are particularly relevant (we follow the original enumeration given in Filloy's list, 1991): (2) conferring intermediate senses; (3) returning to more concrete situations upon the occurrence of an analysis situation; (5) focusing on readings made in language strata that will not allow to solve the problem situation; (8) the presence of inhibitory mechanisms; (9) the presence of obstructions arising from the influence of semantics on syntax and vice versa; (11) the need to confer senses to the networks of ever more abstract actions until they become operations.

For the present study we worked with 12 students from the "Centro Escolar Hermanos Revueltas", in Mexico City. Video-taped clinical interviews were carried out with children who already had been instructed in pre-algebra, and who already had been introduced to topics of elementary algebra through the solution of one-unknown linear equations and of word problems associated to these equations. However, they had not yet been introduced to the systematical use of open algebraic expressions nor of linear equation systems.

The following scheme describes the development process of our research:



INTERVIEW ITEMS

The items list is divided in two sections: word problems and syntactic tasks. The items list proposed to each particular individual may change depending upon the cognitive tendencies found during the interview.

Nine word problems were proposed. The followings are some examples:

> The sum of two numbers equals 90. If one of these numbers is added to 16 and then multiplied by 5 the result is 360. Which are these two numbers?

The difference between two numbers is 27. We know that seven times the minor number plus 30 is 12 times the minor number. Which are these two numbers?

A rectangle's perimeter is five times its width. Its length is 12 meters. ¿What is the width?

The syntax questions were:

S.1	$x + 2 = 4$	**S.6**	$14 + x = 37$	**S.11**	$3x + 4 = 22$	**S.16**	$y - 6 = 3x + 20$
	$x + y = 8$		$4 - y = 28$		$4x + 2y = 34$		$5y - 4x = 64$
S.2	$x + y = 10$	**S.7**	$45 - x = 17$	**S.12**	$3\,\square(8 + x) = 6$	**S.17**	$3x + 8y = 84$
	$x - y = 4$		$x + y = 41$		$2x + y = 23$		$8x + 3y = 59$
S.3	$x + y = 9$	**S.8**	$x + y = 60$	**S.13**	$2\,\square(3 - x) = 6$	**S.18**	$4x - 3 = y$
	$2x + 3y = 23$		$3x = 171$		$4x + 3y = 12$		$6x = y - 7$
S.4	$x + y = 12$	**S.9**	$2\,\square(x + 6) = 84$	**S.14**	$4\,\square(3 - x) = 4$	**S.19**	$3x - 2 = y$
	$5x - 6 = y$		$x + y = 104$		$x + y = 13$		$5x = y + 8$
S.5	$x + 33 = 48$	**S.10**	$4\,\square(x - 8) = 72$	**S.15**	$x - y = 1$		
	$x + y = 73$		$x + y = 17$		$x + y = 5$		

THE EMPIRICAL STUDY: OBSERVATIONS

Because of the space constrains, in this paper we describe two high level performance cases –Mn and Mt– and one average level –L. Through these cases we describe different ways of assigning meaning to new algebraic objects: those obtained through sense production with the comparison and substitution methods.

1. **The "trail and error" strategy**: It appears as a spontaneous solving strategy in all the cases analyzed, and it is linked to the spontaneous readings performed by the learners when they are faced for the first time to two-unknown equation systems. The following is the "strategy" used by one of the cases (L), which illustrates cognitive tendencies 2 and 3:

<table>
<tr><td colspan="2">SL.3. x + y = 9 2x + 3y = 23</td></tr>
<tr><td>L writes: 0 + 9
1 + 8
2 + 7
3 + 6
4 + 5
5 + 4 then

Crosses the last line and then points out line by line, starting at the first one. She stops at "4 + 5" and writes:
4 5
8
15
23

L: The numbers are 4 and 5.
Interviewer: 4 and 5?
L: Yes. First thing I did was to obtain the possible additions which gave 9 as a result and then, by "trail and error": 2 multiplied by 1 is 2 (points out "1 + 8") for checking if they lead to the correct amount.</td><td>Observations: L interprets these two equations as "linked", that is, as equations in which "x" and "y" have the same value in both equations. L writes the different ways for obtaining 9 through the addition of two positive integers and then performs the operations upon the unknowns indicated in the second equation until she finds those values for the unknowns, with which she obtains 23.</td></tr>
</table>

As we will see below these spontaneous readings and strategies can obstruct the learning

of new general solution methods such as comparison and substitution (Cognitive tendencies: 5, 8, 9).

2. **Assigning Sense to Comparison and Substitution and the Meaning of the 'Equal' Sign. Difficulties.** There exist two obstacles which obstruct the application of the comparison method: (A) the reading of objects (unknowns and data) and of operations within the context of positive integers, and (B) a lack of the knowledge required to establish the new equality: the algebraic equivalence. (Tendencies 2, 5, 8, 9).

SMt.17. ***4x □3 = y 6x = y □7***	
Mt wants to obtain the value of '***y***'. So, she transforms the proposed system into: $$\boldsymbol{4x - 3 = y}$$ $$\boldsymbol{6x + 7 = y}$$ but instead of performing the equalization of the two expressions, she looks for the solution through the "trail and error" method using only the positive integers. Mt: In here (***4x - 3 = y***) says that four times '***x***' minus three equals '***y***'. And here (***6x + 7 = y***) says that six '***x***'… plus seven!, equals '***y***'. This (***6x + 7 = y***) has to be bigger than this (***4x - 3 = y***).	*Observations: Mt is able to solve one-unknown equations, regardless of numerical domains of the operated numbers nor of the solutions or the complexity of the equations' algebraic structure.* *Besides, she uses comparison in the case of equation systems derived from verbal problems in which both equations have the same unknown solved and the solutions are positive integers.*

The difficulties related to sense assigning in algebraic substitution are related to the readings of the unknown representation in terms of another unknown. Such readings are performed from different abstraction levels and are accompanied by the inhibition of the use of algebraic substitution. Besides, as expected from the analysis performed with the **Formal Competence Model**, we found that the meaning assigned to the 'equal' sign when used for the equalization of two operation chains which allow the calculation of the same value, is different from the meaning assigned to this sign when algebraic substitution is applied.

3. **Different Abstraction Levels.** The **Communication Model** allows to establish the difference between the readings by the interviewer and the pupil. When a competent user applies the algebraic substitution method he or she uses the equivalence between two expressions knowing the sense of the method, knowing that this method leads him or her to find an unique value for '***x***' and '***y***', if there is any. For example –in the system (S2) that was presented at the introduction, a competent user considers as equivalent the expressions "***4x – 3***" and "***y***", for him or her; these are representations —or names— of the same object: the unknown ***y***. But, when questioned, the learner's spontaneous readings focus on the operations chain performed for calculating the value of the unknown '***y***'. This type of difference in the meanings assigned to algebraic expressions is present in the whole algebra teaching process, in which the teacher is a competent user and the pupils are learners, generating difficulties such as the one described in here. (Tendencies 2 and 5).

4. **The Criss-Cross Method**. Mn invents a solution method which allows him to solve –in combination with the "trail and error" method– all the equation systems proposed. His solution method: ***the criss-cross***, entails the sum of the left member of one of the equations to the right member of the other; the simplification through term elimination and, when possible, and solves the simplified equation for one of the unknowns. This method keeps the equivalence between members as well as the value of the unknowns. Mn is able to perform additions between entire algebraic expressions. The following is an example of how Mn solves an equation system through the combination of *the criss-cross method* and his "trail and error" strategy:

SMn.7. 3x + 8y = 84 8x + 3y = 59
Mn: …There is "***8x + 3y + 84 = 3x + 8y + 59***", so I am mixing them both. If I add this (points towards the left member of the second equation) to this (points towards the right member of the first equation) it is going to be the same as if I added this (the left member of the first equation) to this (the right member of the second equation) because they are both equivalents. If I eliminate from here (***8x + 3y + 84 = 3x + 8y + 59***) "***3x***"… Well, and I also eliminate "***3y***" I get, by the way, that "***5x + 84 = 5y + 59***"… I get "***5x + 25 = 5y***" from which I deduce…Then, if I divide everything by ***5*** I obtain that "***y = x + 5***". From which I deduce that x is minor than y by ***5***… now I have to see, which can the equivalence be? Which is the value of ***x*** and which the value of ***y***? For example, if I say that ***x*** is –just for giving a value– ***4***… No, it can't be ***4*** because suddenly I don't… well if the value of x is four then the value of y is obviously ***9*** and then I get (in ***3x + 8y = 84***) that ***12*** (***3*4***) plus ***9*8*** it is going to be ***72***, ***12 + 72 = 84***, which is satisfactory. Now I will verify with the other (***8x + 3y = 59***), that says ***8x***, that is, ***32*** plus ***27*** (***3*9***) equals ***59***. I check and I think that they are both correct, so I deduce that ***x = 4*** and ***y = 9***.

Mn does not have problems with the numeric domains of the equations and systems that he has to solve. His elimination and simplification strategies are closely linked to the meaning of the 'equal' sign established in a equation. His "trail and error" strategy it is based upon his mental calculation abilities; on his mechanisms for anticipating the numeric values to be obtained; and on his coordination of the actions performed (cognitive tendency 11). In an extensive article we will analyze the potential of ***the criss-cross method*** and the possibility of using it in the regular school teaching processes.

5. **Inhibition of Substitution Usage.** The interview design is directed towards the students' usage of the substitution method, as it can be seen by the last items of the interview in which one of the unknowns is solved for them. However, even Mn –who has enough syntactical competence as to generate a new equalization method: *the criss-cross*– is far from using the substitution method. (Cognitive tendencies 5 and 8).

NEW PERSPECTIVES

This last observation would seem to avail teachers' belief that substitution method's syntax –more difficult to tackle than the comparison method's syntax– is the cause of the *inhibition* presented at observation 5. However, using an ad-hoc formal model for describing each of these methods' syntax, it can be seen that the dialectic between syntax and semantics is the main obstacle in the event of errors when following a rule for which it is necessary to use one or more rules previously and competently used. This is the field in which our present investigation is focusing.

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